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Integrable Symplectic Trilinear Interaction Terms for Matrix Membranes

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Abstract

Cubic interactions are considered in 3 and 7 space dimensions, respectively, for bosonic membranes in Poisson Bracket form. Their symmetries and vacuum configurations are discussed. Their associated first order equations are transformed to Nahm's equations, and are hence seen to be integrable, for the 3-dimensional case, by virtue of the explicit Lax pair provided. The constructions introduced also apply to commutator or Moyal Bracket analogues.

Introduction

A proposal for non-perturbative formulation of M-theory [1] has led to a revival of matrix membrane theory [2, 3]. Symmetry features of membranes and their connection to matrix models [3, 4, 5, 6, 7] have been established for quite some time. Effectively, infinite-N quantum mechanics matrix models presented as a restriction out of SU(∞) Yang-Mills theories amount to membranes, by virtue of the connection between SU(N) and area-preserving diffeomorphisms (*Sdiff*) generated by Poisson Brackets, in which "colour" algebra indices Fourier-transform to "membrane" sheet coordinates. The two are underlain and linked by Moyal Brackets.

Below, we consider novel Poisson Bracket interactions for a bosonic membrane embedded in 3-space

$$\mathcal{L}_{IPB} = \frac{1}{3}\epsilon^{\mu\nu\kappa} X^{\mu}\{X^{\nu},\ X^{\kappa}\}, \tag{1}$$

which are restrictions of the Moyal Bracket generalization

$$\mathcal{L}_{IMB} = \frac{1}{3}\epsilon^{\mu\nu\kappa} X^{\mu}\{\{X^{\nu},\ X^{\kappa}\}\}, \tag{2}$$

which, in turn, also encompasses the plain matrix commutator term

$$\mathcal{L}_{IC} = \frac{1}{3}\epsilon^{\mu\nu\kappa} X^{\mu}[X^{\nu},\ X^{\kappa}]\ . \tag{3}$$

The structure of (1) may be recognized as that of the interaction term $\epsilon_{ijl}\phi^i\epsilon^{\mu\nu}\partial_\mu\phi^j\partial_\nu\phi^j$ of the 2-dimensional SO(3) pseudodual chiral σ-model of Zakharov and Mikhailov [8]—this is a limit [9] of the

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WZWN interaction term, where the integer WZWN coefficient goes to infinity while the coupling goes to zero, such that the product of the integer with the cube of the coupling is kept constant. The structure of (3) is also linked to what remains of the gauge theory instanton density,

$$K^0 = \epsilon^{\mu\nu\kappa}\mathrm{Tr}A_\mu \left(\partial_\nu A_\kappa - \frac{1}{4}[A_\nu, A_\kappa]\right) \ , \tag{4}$$

in the standard space-invariant limit (where the first term vanishes). (N.B. This interaction may be contrasted to the one which appears in a different model [10].) There is some formal resemblance to membrane interaction terms introduced in [11], in that case a quartic in the X^μ's, which, in turn, reflect the symplectic twist of topological terms [12] for self-dual membranes. Unlike those interactions, the cubic terms considered here do not posit full Lorentz invariance beyond 3-rotational invariance: they are merely being considered as quantum mechanical systems with internal symmetry. One may expect this fact to complicate supersymmetrization.

We then also introduce analogous trilinear interactions for membranes embedded in 7-space, which also evince similarly interesting properties.

In what follows, after a brief review of some matrix membrane technology, we discuss the symmetry features of the new terms, the symmetry of the corresponding vacuum configurations, and describe classical configurations of the Nahm type, which we find to be integrable, as in the conventional membrane models. Our discussion will concentrate on Poisson Brackets, but the majority of our results carry over to the Moyal Bracket and commutator cases, by dint of the underlying formal analogy. Subtler considerations of special features for various membrane topologies are not discussed here, and may be addressed as in [13]. Nontrivial boundary terms, e.g. of the type linked to D-branes, are not yet addressed.

Review and Notation

Poisson Brackets, Moyal Brackets, and commutators are inter-related derivative operators, sharing similar properties such as Leibniz chain rules, integration by parts, associativity (so they obey the Jacobi identity), etc. Much of their technology is reviewed in [14, 5, 7, 15].

Poisson Brackets act on the "classical phase-space" of Fourier-transformed colour variables, with membrane coordinates $\xi = \alpha,\ \beta$,

$$\{X^\mu,\ X^\nu\} = \frac{\partial X^\mu}{\partial\alpha}\frac{\partial X^\nu}{\partial\beta} - \frac{\partial X^\mu}{\partial\beta}\frac{\partial X^\nu}{\partial\alpha} \ . \tag{5}$$

This might be effectively regarded as the infinitesimal canonical transformation on the coordinates ξ of X^ν, generated by $\nabla X^\mu \times \nabla$, s.t. $(\alpha,\beta) \mapsto (\alpha - \partial X_\mu/\partial\beta \ ,\ \beta + \partial X_\mu/\partial\alpha)$, which preserves the membrane area element $d\alpha d\beta$. This element is referred to as a *symplectic form* and the class of transformations that leaves it invariant specifies a symplectic geometry; the area preserving diffeomorphisms are dubbed *Sdiff* .

PBs correspond to $N \to \infty$ matrix commutators. But there is a generalization which covers both finite and infinite N. The essentially unique associative generalization of PBs is the Moyal Bracket [14],

$$\{\{X^\mu,\ X^\nu\}\} = \frac{1}{\lambda}\sin\left(\lambda\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta'} - \lambda\frac{\partial}{\partial\beta}\frac{\partial}{\partial\alpha'}\right)\ X^\mu(\xi)X^\nu(\xi')\bigg|_{\xi'=\xi} \ . \tag{6}$$

For $\lambda = 2\pi/N$, ref [5] demonstrates that the Moyal bracket is essentially equivalent to the commutator of SU(N) matrices (or subalgebras of SU(N), depending on the topology of the corresponding membrane surface involved in the Fourier-transform of the colour indices [7, 13]). In the limit $\lambda \to 0$, the Moyal bracket goes to the PB (i.e. λ may be thought of as $\hbar$). Thus, PBs are seen to represent the infinite N limit. This type of identification was first noted by Hoppe on a spherical membrane surface [3]; the foregoing Moyal limit argument was first formulated on the torus [5], but extends naturally to other topologies [5, 7, 13].

Refs [4] utilize the abovementioned identification of SU(∞) with *Sdiff* on a 2-sphere, to take the large N limit of SU(N) gauge theory and produce membranes. This procedure was found to be more transparent

on the torus [7]: the Lie algebra indices Fourier-conjugate to surface coordinates, and the fields are rescaled Fourier transforms of the original SU(N) fields. The group composition rule for them is given by the PBs and the group trace by surface integration,

$$[A_\mu, A_\nu] \ \mapsto \{a_\mu, a_\nu\} \ ; \tag{7}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ \mapsto f_{\mu\nu}(\alpha,\beta) = \partial_\mu a_\nu - \partial_\nu a_\mu + \{a_\mu, a_\nu\} \ ; \tag{8}$$

$$\mathrm{Tr} F_{\mu\nu} F_{\mu\nu} \mapsto -\frac{N^3}{64\pi^4} \int d\alpha d\beta \ f_{\mu\nu}(\alpha,\beta) \ f_{\mu\nu}(\alpha,\beta). \tag{9}$$

But the large N limit need not be taken to produce sheet actions. The Lagrangian with the Moyal Bracket supplanting the Poisson Bracket is itself a gauge-invariant theory, provided that the gauge transformation also involves the Moyal instead of the Poisson bracket:

$$\delta a_\mu = \partial_\mu \Lambda - \{\{\Lambda, a_\mu\}\} \ , \tag{10}$$

and hence, by virtue of the Jacobi identity,

$$\delta f_{\mu\nu} = -\{\{\Lambda, f_{\mu\nu}\}\} \ . \tag{11}$$

colour invariance then follows,

$$\delta \int d\alpha d\beta \ f_{\mu\nu} f_{\mu\nu} = -2 \int d\alpha d\beta \ f_{\mu\nu} \{\{\Lambda, f_{\mu\nu}\}\} = 0 \ . \tag{12}$$

The relevant manipulations are specified in [7], and the last equality is evident by integrations by parts, where the surface term is discarded, or nonexistent if the colour membrane surface is closed. (But note this topological term may be nontrivial for D-membranes.) For $\lambda = 2\pi/N$, this is equivalent to the conventional SU(N) commutator gauge theory.

Consider the SU(∞) Yang Mills Lagrangian and dimensionally reduce all space dependence [3], leaving only time dependence, while preserving all the colour-Fourier-space (membrane coordinates $\xi = \alpha, \ \beta$) dependence of the gauge fields, which are now denoted $X^\mu(t, \alpha, \beta)$. Fix the gauge to $X^0 = 0$. The Yang-Mills lagrangian density reduces to the bosonic membrane lagrangian density

$$\mathcal{L}_{PB} = \frac{1}{2} (\partial_t X^\mu)^2 - \frac{1}{4} \{X^\mu, \ X^\nu\}^2. \tag{13}$$

The PB is also the determinant of the tangents to the membrane, so the conventional "potential term" was identified in [7] as the Schild-Eguchi string lagrangian density [16] (sheet area squared instead of area), $\{X_\mu, X_\nu\}\{X_\mu, X_\nu\}$. (It can be seen that the equations of motion of such a string action contain those of Nambu's action.)

Note that, fixing the gauge $X_0 = 0$ preserves the global colour invariance, i.e. with a time-independent parameter $\Lambda(\alpha,\beta)$. The action is then invariant under

$$\delta X^\mu = \{\Lambda, \ X^\mu\}. \tag{14}$$

By Noether's theorem, this implies the time invariance of the colour charge,

$$\mathcal{Q}_\Lambda = \int d\alpha d\beta \ \Lambda(\alpha,\beta) \ \{\partial_t X^\mu, \ X^\mu\}. \tag{15}$$

The same also works for the Moyal case [7]. The corresponding Moyal Schild-Eguchi term was utilized to yield a "star-product-membrane" [15],

$$\mathcal{L}_{MB} = \frac{1}{2} (\partial_t X^\mu)^2 - \frac{1}{4} \{\{X^\mu, \ X^\nu\}\}^2, \tag{16}$$

invariant under

$$\delta X^\mu = \{\{\Lambda,\ X^\mu\}\}. \tag{17}$$

As outlined, this includes the commutator case,

$$\mathcal{L}_C = \frac{1}{2}(\partial_t X^\mu)^2 - \frac{1}{4}[X^\mu,\ X^\nu]^2, \tag{18}$$

invariant under

$$\delta X^\mu = [\Lambda,\ X^\mu]. \tag{19}$$

Discussion of the Cubic Terms for 3 Dimensions

By suitable integrations by parts, it is straightforward to check that the cubic terms (1,2,3) in the respective actions, $\int dt d\alpha d\beta\ \mathcal{L}$, are 3-rotational invariant, as well as time-translation invariant and translation symmetric. They are also global colour invariant, as specified above.

Let us also consider a plain mass term in the action, (of the type that may arise as a remnant of space gradients in compactified dimensions),

$$\mathcal{L}_{3dPB} = \frac{1}{2}(\partial_t X^\mu)^2 - \frac{1}{4}\{X^\mu,\ X^\nu\}^2 - \frac{m}{2}\epsilon^{\mu\nu\kappa} X^\mu \{X^\nu,\ X^\kappa\} - \frac{m^2}{2}(X^\mu)^2. \tag{20}$$

The second order equation of motion,

$$\partial_t^2 X^\mu = -m^2 X^\mu - \frac{3m}{2}\epsilon^{\mu\nu\kappa}\{X^\nu,\ X^\kappa\} - \{X^\nu, \{X^\mu,\ X^\nu\}\}, \tag{21}$$

follows not only from extremizing the action, but also results from a first-order equation of the Nahm (self-dual) type [17], albeit complex,

$$\partial_t X^\mu = im X^\mu + \frac{i}{2}\epsilon^{\mu\nu\kappa}\{X^\nu,\ X^\kappa\}. \tag{22}$$

These equations hold for PBs, MBs, and commutators.

For solutions of this first-order equation, the conserved energy vanishes. In general, however, such solutions are not real, and do not provide absolute minima for the action. Nonetheless, the lagrangian density can be expressed as a sum of evocative squares, since the potential in (20) is such a sum,

$$\mathcal{L}_{3dPB} = \frac{1}{2}(\partial_t X^\mu)^2 - \frac{1}{2}\left(m X^\mu + \frac{1}{2}\epsilon^{\mu\nu\kappa}\{X^\nu,\ X^\kappa\}\right)^2. \tag{23}$$

By integration by parts in the action $\int dt d\alpha d\beta\ \mathcal{L}_{3dPB}$, the lagrangian density itself can then be altered to

$$\mathcal{L}_{3dPB} \cong -\frac{1}{2}\left(i\partial_t X^\mu + m X^\mu + \frac{1}{2}\epsilon^{\mu\nu\kappa}\{X^\nu,\ X^\kappa\}\right)^2, \tag{24}$$

just like the conventional bosonic membrane lagrangian density (the congruence symbol, $\cong$, indicates equivalence up to surface terms, which, e.g., vanish for a closed surface; again, consideration of D-membranes would proceed separately). The complex-conjugate versions of the above are equally valid.

Vacuum Configurations

The minimum of the conventional matrix membrane trough potential favors alignment of the X^μs. The mass parameter introduced above parameterizes a partial trough symmetry breaking [18], but does not lift "dilation" invariance, seen as follows.

The static (t-independence) minima for the action (vacuum configurations) are solutions of

$$m X^\mu + \frac{1}{2}\epsilon^{\mu\nu\kappa}\{X^\nu,\ X^\kappa\} = 0\ . \tag{25}$$

The previously considered case, $m = 0$, is easily solved by "colour-parallel" configurations. But for $m \neq 0$, the static solutions must lie on a 2-sphere, since from the previous equation

$$X^\mu \frac{\partial X^\mu}{\partial \alpha} = 0 = X^\mu \frac{\partial X^\mu}{\partial \beta} \ , \tag{26}$$

so

$$X^\mu X^\mu = R^2 \ , \tag{27}$$

an unspecified constant. (Hence $\epsilon^{\mu\nu\kappa} X^\mu \{X^\nu,\ X^\kappa\} = -2mR^2$.) However, from (25), note that both m and also R, the scale of the X^μs, can be absorbed in the membrane coordinates ξ and will not be specified by the solution of (25).

Indeed, solving for one coordinate component on this sphere, say

$$X(Y,Z) = \pm\sqrt{R^2 - Y^2 - Z^2} \ , \tag{28}$$

reduces the three equations (25) to one. Namely,

$$\{Z,Y\} = m\sqrt{R^2 - Y^2 - Z^2} \ , \tag{29}$$

on the positive X branch ($m \mapsto -m$ on the negative X branch). This last equation is solved by

$$Z = \alpha, \qquad Y = \sqrt{R^2 - \alpha^2}\sin(m\beta) \ . \tag{30}$$

One can then interpret $m\beta$ as the usual azimuthal angle around the Z-axis. Hence, $-\pi/2 \leq m\beta \leq \pi/2$ and $-R \leq \alpha \leq R$ covers the $X \geq 0$ hemisphere completely. The other hemisphere is covered completely by the negative X branch. Since R is not fixed, it amounts to an unlifted residual trough dilation degeneracy.

All static solutions are connected to this explicit one by rescaling R and exploiting the equation's area-preserving diffeomorphism invariance for $\xi = (\alpha, \beta)$.

Nahm's Equation and Lax Pairs

The first order equation, (22), simplifies upon changing variables to $\tau = \mathrm{e}^{\mathrm{imt}}/\mathrm{m}$ and $X^\mu = \mathrm{e}^{\mathrm{imt}}\mathrm{Y}^\mu$, and reduces to the conventional PB version [19] of Nahm's equation [17],

$$\partial_\tau Y^\mu = \frac{1}{2}\epsilon^{\mu\nu\kappa}\{Y^\nu,\ Y^\kappa\}. \tag{31}$$

This has real solutions, and can be linearized by Ward's transformation [19]. However, the action (23) does not reduce to the conventional one upon these transformations. (Likewise, the second order equations of motion only reduce to $\partial_\tau^2 Y^\mu = \{Y^\nu, \{Y^\mu, Y^\nu\}\} + \frac{3}{\tau}(\partial_\tau Y^\mu - \frac{\epsilon^{\mu\nu\kappa}}{2}\{Y^\nu, Y^\kappa\})$.)

Moreover, note

$$\partial_\tau Y^\mu \partial_\tau Y^\mu = \frac{\partial(Y^1, Y^2, Y^3)}{\partial(\tau, \alpha, \beta)}, \tag{32}$$

$$\partial_\tau Y^\mu \partial_\xi Y^\mu = 0. \tag{33}$$

One may further utilize the cube root of unity, $\omega = \exp(2\pi i/3)$, (note $\omega(\omega-1)$ is pure imaginary) to recast (31),

$$L \equiv \omega Y^1 + \omega^2 Y^2 + Y^3 \ , \qquad \overline{L} = \omega^2 Y^1 + \omega Y^2 + Y^3 \ , \qquad M \equiv Y^1 + Y^2 + Y^3 \ , \tag{34}$$

$$\omega(\omega-1)\ \partial_\tau L = \{M\ ,\ L\} \ , \qquad \omega(\omega-1)\ \partial_\tau \overline{L} = -\{M\ ,\ \overline{L}\} \ , \qquad \omega(\omega-1)\ \partial_\tau M = \{L\ ,\overline{L}\} \ , \tag{35}$$

which thus yields an infinite number of complex time-invariants,

$$Q_n = \int d\alpha d\beta\ L^n \ , \tag{36}$$

for arbitrary integer power n, as the time derivative of the integrand is a surface term. (This is in complete analogy with the standard case of commutators.) This is linked to classical integrability, as discussed next.

Eqs (35) amount to one complex and one real equation, but these can further be compacted into just one by virtue of an arbitrary real spectral parameter ζ, introduced in [20]:

$$H \equiv \frac{i}{\sqrt{2}\omega(\omega-1)}\left(\zeta L - \frac{\overline{L}}{\zeta}\right) , \qquad K \equiv i\sqrt{2}M + \zeta L + \frac{\overline{L}}{\zeta} , \tag{37}$$

$$\partial_\tau K = \{H, K\} . \tag{38}$$

(Note the wave solution $H = \alpha,\ L = f(\beta+\tau)$.) This Lax pair, analogous to [20], likewise leads to a one-parameter family of time-invariants,

$$\mathcal{Q}_n(\zeta) = \int d\alpha d\beta\ K^n , \tag{39}$$

and a Lax isospectral flow,

$$\mathcal{K} \equiv \nabla K \times \nabla, \qquad \mathcal{H} \equiv \nabla H \times \nabla, \qquad \nabla \equiv \left(\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}\right) , \tag{40}$$

such that:

$$\partial_\tau \mathcal{K} = \mathcal{H}\mathcal{K} - \mathcal{K}\mathcal{H} . \tag{41}$$

As a consequence, the spectrum of $\mathcal{K}$ is preserved upon time evolution by the (pure imaginary) $\mathcal{H}$:

$$\partial_\tau \psi = \mathcal{H}\psi, \tag{42}$$

since time-differentiating

$$\mathcal{K}\psi = \lambda\psi \tag{43}$$

and applying the above yields

$$(\partial_\tau\lambda)\ \psi = (\partial_\tau\mathcal{K})\psi + \mathcal{K}\partial_\tau\psi - \lambda\ \partial_\tau\psi = 0 . \tag{44}$$

This isospectral flow then provides integrability [8] for (22), as in the case of the conventional Nahm equation.

This discussion also carries over to plain commutators or Moyal Brackets as well, with suitable adaptations [5].

Duality and Implicit Solution of the (PB) Nahm Equation

Ward [19] has solved eq (31) implicitly through twistor linearization.

Another solution procedure for the construction of a wide class of solutions to (31) may be found by interchanging the rôles of dependent and independent variables; the equations then take the form

$$\frac{\partial\tau}{\partial Y^1} = \frac{\partial\alpha}{\partial Y^2}\frac{\partial\beta}{\partial Y^3} - \frac{\partial\alpha}{\partial Y^3}\frac{\partial\beta}{\partial Y^2} , \tag{45}$$

together with cyclic permutations, i.e.

$$\partial_\mu\tau = \epsilon^{\mu\nu\rho}\partial_\nu\alpha\ \partial_\rho\beta . \tag{46}$$

Cross-differentiation produces integrability conditions

$$\partial_\mu(\partial_\kappa\alpha\ \partial_\mu\beta - \partial_\mu\alpha\ \partial_\kappa\beta) = 0. \tag{47}$$

Another evident consistency condition is

$$\partial^2 \tau = 0 \ , \tag{48}$$

but this will be guaranteed by the integrability of those equations. Having found α, β in terms of Y^1, Y^2, Y^3, the solution for τ follows by quadratures.

Evidently, solutions of these dual equations

$$\partial_\mu f = \epsilon^{\mu\nu\rho} \partial_\nu g \ \partial_\rho h \ , \tag{49}$$

for $f(Y^1, Y^2, Y^3)$ produce constants of the motion $\int d^2\xi \ f$, beyond those already found by the Lax procedure, for the original equation (31), in illustration of a phenomenon noted in [21], as it is straightforward to verify that $df/d\tau = \{g, h\}$. Actually, f need only solve Laplace's equation (48): any harmonic function $f(Y^1, Y^2, Y^3)$ yields a conserved density for (31), by also satisfying (49). By virtue of Helmholtz's theorem, a divergenceless 3-vector ∇f is representable as a curl of another vector $\mathbf{A}$. On the other hand, an arbitrary 3-vector can also be represented in terms of three scalars by means of the Clebsch decomposition of that vector as $\mathbf{A} = g\nabla h + \nabla u$. In fact, the problem of solving the inverse Nahm equation (45) is equivalent to determining the Clebsch decomposition of an arbitrary vector.

A class of solutions can be found by postulating a simple dependence on Y^3; with the Ansatz

$$\tau = f(Y^1, Y^2), \quad \alpha = \mathrm{e}^{-\mathrm{mY}^3}\mathrm{g}(\mathrm{Y}^1, \mathrm{Y}^2), \quad \beta = \mathrm{e}^{\mathrm{mY}^3}\mathrm{h}(\mathrm{Y}^1, \mathrm{Y}^2), \tag{50}$$

it is seen that the equations are satisfied, provided g is an arbitrary function of h,

$$g = \phi(h), \tag{51}$$

as well as

$$\tau = \Re F(Y^1 + iY^2), \quad mgh = \Im F(Y^1 + iY^2) \ , \tag{52}$$

for an arbitrary analytic function F, since these combinations must satisfy the Cauchy-Riemann conditions.

Membrane Embedding in 7 Dimensions

Remarkably, the same type of term may also be introduced for a membrane embedded in 7 space dimensions. A self-dual (antisymmetric) 4-tensor in 8 dimensions, $f_{\mu\nu\rho\sigma}$ was invoked [22] as an 8-dimensional analogue of the 4-dimensional fully antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$. Some useful technology for the manipulation of this tensor (which has 35 nonzero components and is invariant under a particular $SO(7)$ subgroup of $SO(8)$) can also be found in [23]; in particular, the identity

$$f^{0\mu\nu\kappa} f^{0\mu\lambda\rho} = f^{\nu\kappa\lambda\rho} + \delta^{\nu\lambda}\delta^{\kappa\rho} - \delta^{\kappa\lambda}\delta^{\nu\rho} \ . \tag{53}$$

By analogy with (31), one can postulate a first order equation

$$\partial_\tau Y^\mu - \frac{f^{0\mu\nu\kappa}}{2}\{Y^\nu, \ Y^\kappa\} = 0 \ . \tag{54}$$

The indices run from $\mu = 1$ to $\mu = 7$, since we are working in a gauge where $Y^0 = 0$. The second order equation arising from iteration of (54), by virtue of the above identity as well as the Jacobi identity is

$$\partial_{\tau\tau} Y^\mu = -\{\{Y^\mu, \ Y^\nu\}, \ Y^\nu\} \ . \tag{55}$$

This arises from the lagrangian density

$$\mathcal{L}_{7dPB} = \frac{1}{2}(\partial_\tau Y^\mu)^2 + \frac{1}{4}\{Y^\mu, \ Y^\nu\}^2 \ . \tag{56}$$

As in the 3-dimensional case, this action is a sum of squares up to a mere surface term,

$$\frac{1}{2}\left(\partial_\tau Y^\mu - \frac{f^{0\mu\nu\kappa}}{2}\{Y^\nu,\ Y^\kappa\}\right)^2 = \mathcal{L}_{7dPB} - f^{0\mu\nu\kappa}\partial_\tau Y^\mu \partial_\alpha Y^\nu \partial_\beta Y^\kappa \cong \mathcal{L}_{7dPB}. \tag{57}$$

Apparent extra quartic terms in this lagrangian density have, in fact, vanished by virtue of the identity,

$$\{f,\ g\}\{h,\ k\} + \{f,\ h\}\{k,\ g\} + \{f,\ k\}\{g,\ h\} \equiv 0, \tag{58}$$

which holds for Poisson Brackets on a 2-dimensional phase-space—but not for commutators nor Moyal Brackets[2]. This cancellation works at the level of the lagrangian density for the PB case. However, note that even for ordinary matrices the corresponding term would vanish in the traced action, by the cyclicity of the trace pitted against full antisymmetry,

$$f^{\mu\nu\kappa\rho}\mathrm{Tr}X^\mu X^\nu X^\kappa X^\rho = 0\ . \tag{59}$$

Likewise, the corresponding interaction for Moyal Brackets,

$$f^{\mu\nu\kappa\rho}\int d^2\xi\ \{\{X^\mu, X^\nu\}\}\{\{X^\kappa, X^\rho\}\}, \tag{60}$$

is forced by associativity to reduce to a surface term, vanishing unless there are contributions from surface boundaries or D-membrane topological numbers involved. (Shortcuts for the manipulation of such expressions underlain by $\star$-products can be found in, e.g. [15].) The cross terms involving time derivatives are expressible as divergences, as in the 3-dimensional case, and hence give rise to possible topological contributions.

As a result, (54) is the Bogomol'nyi minimum of the action (56) with the bottomless potential.

As in the case of 3-space, the conventional membrane signs can now be considered (for energy bounded below), and a symmetry breaking term m introduced, to yield

$$\mathcal{L}_{7dPB} \cong -\frac{1}{2}\left(-i\partial_t X^\mu + mX^\mu + \frac{f^{0\mu\nu\kappa}}{2}\{X^\nu,\ X^\kappa\}\right)^2 . \tag{61}$$

This model likewise has 7-space rotational invariance, and its vacuum configurations are, correspondingly, 2-surfaces lying on the spatial 6-sphere embedded in 7-space: $X^\mu X^\mu = R^2$. But, in addition, because of (53), these surfaces on the sphere also satisfy the trilinear constraint

$$f^{\lambda\mu\nu\kappa}X^\mu\{X^\nu,\ X^\kappa\} = 0\ , \tag{62}$$

for $\lambda \neq 0$. (For $\lambda = 0$ this trilinear is $-2mR^2$.)

Higher-dimensional Analogues

The system of equations (54) does not appear to readily yield integrability properties. However, it may be extended to a 9-dimensional system—or, equivalently, a 10-dimensional one if $\partial_\tau Y^\mu$ is replaced by $\{Y^0,\ Y^\mu\}$. This different system, though it breaks the 9-dimensional rotational invariance, is integrable by virtue of the additional six constraints imposed on the system.

Writing out the system (54) explicitly and augmenting it with two more variables, $Y^8,\ Y^9$, gives rise to the equations

[2] D. Fairlie and A. Sudbery, 1988, unpublished. It follows from $\epsilon^{[jk}\delta^{l]m} = 0$, whence $\epsilon^{[jk}\epsilon^{l]m} = 0$ for these membrane symplectic coordinates.

$$\begin{aligned}
\partial_\tau Y^1 &= \{Y^2,\ Y^3\} + \{Y^6,\ Y^5\} + \{Y^4,\ Y^7\} + \{Y^8,\ Y^9\} \\
\partial_\tau Y^2 &= \{Y^3,\ Y^1\} + \{Y^4,\ Y^6\} + \{Y^5,\ Y^7\} \\
\partial_\tau Y^3 &= \{Y^1,\ Y^2\} + \{Y^5,\ Y^4\} + \{Y^6,\ Y^7\} \\
\partial_\tau Y^4 &= \{Y^3,\ Y^5\} + \{Y^6,\ Y^2\} + \{Y^7,\ Y^1\} \\
\partial_\tau Y^5 &= \{Y^4,\ Y^3\} + \{Y^1,\ Y^6\} + \{Y^7,\ Y^2\} \\
\partial_\tau Y^6 &= \{Y^2,\ Y^4\} + \{Y^5,\ Y^1\} + \{Y^7,\ Y^3\} \\
\partial_\tau Y^7 &= \{Y^1,\ Y^4\} + \{Y^2,\ Y^5\} + \{Y^3,\ Y^6\} \\
\partial_\tau Y^8 &= \{Y^9,\ Y^1\} \\
\partial_\tau Y^9 &= \{Y^1,\ Y^8\} \\
0 &= \{Y^8,\ Y^2\} + \{Y^3,\ Y^9\} \\
0 &= \{Y^8,\ Y^3\} + \{Y^9,\ Y^2\} \\
0 &= \{Y^8,\ Y^6\} + \{Y^5,\ Y^9\} \\
0 &= \{Y^8,\ Y^5\} + \{Y^9,\ Y^6\} \\
0 &= \{Y^8,\ Y^4\} + \{Y^7,\ Y^9\} \\
0 &= \{Y^8,\ Y^7\} + \{Y^9,\ Y^4\}\ .
\end{aligned} \tag{63}$$

The sums of the squares of these expressions amount to the sums of the squares of the individual terms, each appearing once and once only, with unit coefficient, the cross terms vanishing by the identity (58). Other schemes are also possible [24], but in the 10-dimensional cases investigated [25], the quadratic terms do not arise with both unit multiplicity and positive sign. Whilst the cases considered in the latter reference were directly associated with the reduction of $N = 1$ supersymmetric 10-dimensional Yang Mills to the $N = 4$ 4-dimensional case, the symmetry breakdown in (63) does not follow this pattern.

The terms not involving a τ-derivative in (63) may be rearranged as follows,

$$\{(Y^2 \pm iY^3),\ (Y^8 \pm iY^9)\} = \{(Y^6 \pm iY^5),\ (Y^8 \pm iY^9)\} = \{(Y^4 \pm iY^7),\ (Y^8 \pm iY^9)\} = 0. \tag{64}$$

This means that these "commuting" combinations of variables are functionally related:

$$(Y^2 + iY^3) = A_+(\alpha, \beta, \tau),\quad (Y^6 + iY^5) = B_+(A_+),\quad (Y^8 + iY^9) = C_+(A_+),\quad (Y^4 + iY^7) = D_+(A_+), \tag{65}$$

with a corresponding result for the negative combinations. This implies that all τ derivative equations, apart from the first, are equivalent to the Lax pair combinations

$$\partial_\tau (Y^2 \pm iY^3) = \pm i\{Y^1,\ (Y^2 \pm iY^3)\}, \tag{66}$$

i.e. the 6 constraints have reduced 9 equations to 3. The arbitrariness in the functions $B_\pm, C_\pm,\ D_\pm$, implies that the first equation in (63) is less restrictive than the corresponding case of (31). Thus (63) are a fortiori integrable, and the solution is not dissimilar to the 3-dimensional case.

Even though mostly integrable first-order equations have been studied in this note, it should be borne in mind that the behaviour of the generic solutions to the second-order equations of motion for such systems is often chaotic. For example, in the case of Yang-Mills with a finite gauge group, with fields dependent only upon time, (18), characteristic features of chaotic behaviour have been demonstrated for the solutions of the second-order equations of motion [26]. It would be interesting to also know what is the situation for the behaviour of solutions to the second-order dynamical equations of the PB system (13).

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